

EULER MACLAURIN WITH REMAINDER FOR A SIMPLE INTEGRAL POLYTOPE

Yael Karshon, Shlomo Sternberg, and Jonathan Weitsman

ABSTRACT. We give an Euler Maclaurin formula with remainder for the sum of the values of a smooth function on the integral points in a simple integral polytope. We prove this formula by elementary methods.

1. INTRODUCTION

The Euler Maclaurin formula computes the sum of the values of a function f over the integer points in an interval in terms of the integral of f over variations of that interval. A version of this classical formula is this:

For any function $f(x)$ on the real line and any integers $a < b$, we will consider the weighted sum

$$(1) \quad \sum_{[a,b]}' f := \frac{1}{2}f(a) + f(a+1) + \dots + f(b-1) + \frac{1}{2}f(b).$$

If f is “nice enough”, for instance, a polynomial, then

$$(2) \quad \sum_{[a,b]}' f = \mathbf{L}\left(\frac{\partial}{\partial h_1}\right)\mathbf{L}\left(\frac{\partial}{\partial h_2}\right) \int_{a-h_1}^{b+h_2} f(x)dx \Big|_{h_1=h_2=0},$$

where

$$(3) \quad \mathbf{L}(S) = \frac{S/2}{\tanh(S/2)} = 1 + \sum_{k=1}^{\infty} \frac{1}{(2k)!} b_{2k} S^{2k}.$$

Because $\int_{a-h_1}^{b+h_2} f(x)dx$ is a polynomial in h_1 and h_2 if f is a polynomial in x , applying the infinite order differential operator $\mathbf{L}(\frac{\partial}{\partial h_i})$ then yields a finite sum, so the right hand side of (2) is well defined when f is a polynomial.

A polytope in \mathbb{R}^n is called *integral*, or a *lattice polytope*, if its vertices are in the lattice \mathbb{Z}^n ; it is called *simple* if exactly n edges emanate from each vertex; it is called *regular* if, additionally, the edges emanating from each vertex lie along lines which are generated by a \mathbb{Z} -basis of the lattice \mathbb{Z}^n .

Khovanskii and Pukhlikov [KP1, KP2], following Khovanskii [Kh1, Kh2], generalized the classical Euler Maclaurin formula to give a formula for the sums of the values of polynomial or exponential functions on the lattice points in higher dimensional convex polytopes Δ which are integral and regular. This formula was generalized to *simple* integral polytopes by Cappell and Shaneson [CS1, CS2, CS3, S], and subsequently by Guillemin [Gu2] and by Brion-Vergne [BV]. All of these generalizations involve “corrections” to the Khovanskii-Pukhlikov formula when the simple polytope is not regular. When applied to the constant function $f \equiv 1$, these Euler Maclaurin formulas compute the number of lattice points in Δ in terms of the volumes of “dilations” of Δ . A small

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sample of the literature on the problem of counting lattice points in convex polytopes is given in [Pi, Md, V, KK, Mo, Pom, DR, BDR, Hat]; see the survey [BP] and references therein.

These formulas are closely related to the Riemann-Roch formula from algebraic geometry via the correspondence between polytopes and toric varieties. Under this correspondence, regular polytopes correspond to smooth toric varieties, and the Khovanskii-Pukhlikov formula was motivated in this way, although it was proved combinatorially. Cappell and Shaneson derived their formula from their theory of characteristic classes of singular algebraic varieties and had the key idea of using the operator \mathbf{L} as we do here. Guillemin obtained his formula from the equivariant Kawasaki-Riemann-Roch formula and methods coming from symplectic geometry and the theory of geometric quantization. Brion and Vergne employed a method that is closer to that in the original proof of Khovanskii and Pukhlikov, using Fourier analysis.

To illustrate the relation to toric varieties, let us sketch the symplectic-geometric proof for the case of a regular polytope, following Guillemin [Gu1]. This approach will not be used elsewhere in this paper. A regular integral polytope $\Delta \subset \mathbb{R}^n$ determines a smooth Kähler toric variety (M, ω) , and geometric quantization gives rise to a virtual representation $Q(M)$ of the torus T^n . The dimension $\dim Q(M)$ of this quantization is equal to the number of lattice points in Δ . (This result (see [Od, Corollary 2.23]) is an expression of the “quantization commutes with reduction” principle in symplectic geometry [GS]. According to this principle, $\dim Q(M)^c = \dim Q(M_c)$ for each lattice point $c \in \mathbb{Z}^n \subset \text{Lie}(T^n)^*$, where $Q(M)^c$ is the subspace of $Q(M)$ on which T^n acts through the character given by c , and where M_c is the reduced space of M at c . Because M is a toric variety, M_c is a point if $c \in \Delta$ and is empty otherwise.) On the other hand, by the Hirzebruch-Atiyah-Singer generalization of the classical Riemann-Roch formula, we have $\dim Q(M) = \int_M \exp(c_1(L)) \text{Td}(TM)$, where $c_1(L) = [\omega]$ is the Chern class of the pre-quantization line bundle and $\text{Td}(TM)$ is the Todd class of the tangent bundle. Expressing M as a reduction of a linear torus action on \mathbb{C}^d (where d is the number of facets of Δ), the tangent bundle stably splits into line bundles L_1, \dots, L_d , and the above integral is obtained by applying the Khovanskii-Pukhlikov differential operator $\prod \text{Td}(\frac{\partial}{\partial h_i})$ to the integral $\int_M \exp(\omega + \sum h_i c_1(L_i))$. The Duistermaat-Heckman theorem on the variation of reduced symplectic structures implies that this integral is equal to the volume of the polytope $\Delta(h)$ that is obtained from Δ by shifting the i th facet by a distance h_i , for $i = 1, \dots, d$. Hence, the number of lattice points in Δ is obtained by applying the Khovanskii-Pukhlikov operator to the volume of $\Delta(h)$.

The Euler-Maclaurin formulas due to Khovanskii-Pukhlikov, Cappell-Shaneson, Guillemin, and Brion-Vergne are all exact formulas, valid for sums of exponential or polynomial functions. Cappell and Shaneson [CS3] have also investigated the problem of deriving an Euler-Maclaurin formula with remainder. In a previous paper [KSW], we stated and proved an Euler-Maclaurin formula with remainder for the sum of the values of an arbitrary smooth function on the lattice points in a *regular* polytope, and adumbrated a generalization to the case of simple integral polytopes. The purpose of this paper is to state and prove an Euler-Maclaurin formula with remainder for simple lattice polytopes (Theorem 2). The key ingredients in the proof of this theorem, as in [KSW], are a variant of the Euler-Maclaurin formula in one dimension, given in Proposition 27, which by iteration gives a formula for *orthants*, along with a combinatorial result, given in Proposition 38, which shows how the sum of the values of a function over the lattice points in a polytope can be decomposed into sums over such orthants.

Our Euler-Maclaurin formula with remainder is stated in Theorem 2 for functions of compact support. In Section 7 we show how to extend it to symbols (in the sense of Hormander, see e.g. [Ho]). This is the content of Theorem 3. As a corollary, we deduce an exact Euler-Maclaurin formula for polynomials.

The early references to the Euler-Maclaurin formula are Euler [Eu] and Maclaurin [Ma]. Apparently, Poisson [Poi] was the first to give a remainder formula. See also [Har].

2. WEIGHTED SUMS IN ONE DIMENSION

Exact Euler Maclaurin. Here is a brief proof of the exact Euler Maclaurin formula (2); cf. [BV]. First, we prove this formula when $f(x)$ is an exponential function: $f(x) = e^{\lambda x}$ with $|\lambda| < 2\pi$. The formula then becomes

$$(4) \quad \sum_{[a,b]}' e^{\lambda x} = \mathbf{L}\left(\frac{\partial}{\partial h_1}\right) \mathbf{L}\left(\frac{\partial}{\partial h_2}\right) \int_{a-h_1}^{b+h_2} e^{\lambda x} dx \Big|_{h_1=h_2=0}.$$

An explicit computation, which uses the facts that the constant term in the formal power series $\mathbf{L}(S)$ is one and that $\mathbf{L}(-S) = \mathbf{L}(S)$, shows that

$$(5) \quad \mathbf{L}^{2k}\left(\frac{\partial}{\partial h_1}\right) \mathbf{L}^{2k}\left(\frac{\partial}{\partial h_2}\right) \int_{a-h_1}^{b+h_2} e^{\lambda x} dx \Big|_{h_1=h_2=0} = \mathbf{L}^{2k}(\lambda) \int_a^b e^{\lambda x} dx$$

where $\mathbf{L}^{2k}(S)$ is the truncation of the power series $\mathbf{L}(S)$ at the even integer $2k$. The radius of convergence of the power series $\mathbf{L}(\lambda)$ is 2π because the zeros of $\tanh(\lambda/2)$ that are nearest to the origin are at $\pm 2\pi i$. Hence, for $|\lambda| < 2\pi$, the expression in (5) converges as $k \rightarrow \infty$ to

$$(6) \quad \mathbf{L}(\lambda) \int_a^b e^{\lambda x} dx = \frac{\lambda/2}{\tanh(\lambda/2)} \left(\frac{e^{\lambda b}}{\lambda} - \frac{e^{\lambda a}}{\lambda} \right) = \frac{1}{2} \left(e^{\lambda/2} + e^{-\lambda/2} \right) \frac{e^{\lambda b} - e^{\lambda a}}{e^{\lambda/2} - e^{-\lambda/2}},$$

which is equal to the left hand side of (4) by the formula for a geometric sum.

Moreover, this convergence is uniform on any closed sub-disk of $|\lambda| < 2\pi$, because $\mathbf{L}(\lambda)$ is a power series and $\int_a^b e^{\lambda x} dx$ is bounded away from 0 and from ∞ . Recall that differentiation commutes with uniform limits of holomorphic functions (as a consequence of the Cauchy formula). It follows that the derivative $\frac{\partial}{\partial \lambda}$ commutes with the infinite order differential operators $\mathbf{L}(\frac{\partial}{\partial h_i})$ on the right hand side of (4). Comparing the Taylor coefficients of λ^n on the left and right hand sides of (4), we get a similar formula for $f(x) = x^n$, and hence for all polynomials.

Remark 7. In higher dimensions, the problem of obtaining a formula for polynomials from a formula for exponentials is addressed in [KP2].

Weighted sums and ordinary sums. The Todd function is defined by

$$\text{Td}(S) = \frac{S}{1 - e^{-S}}.$$

The Bernoulli numbers are the coefficients b_k in

$$\text{Td}(-S) = \frac{S}{e^S - 1} = 1 + \sum_{k \geq 1} \frac{1}{k!} b_k S^k.$$

(We are following the conventions in Bourbaki [Bo].) Since

$$\begin{aligned} \mathbf{L}(S) &= (S/2) \frac{e^{S/2} + e^{-S/2}}{e^{S/2} - e^{-S/2}} = \frac{1}{2} \left(\frac{S}{1 - e^{-S}} + \frac{S}{e^S - 1} \right) \\ &= \frac{1}{2} (\text{Td}(S) + \text{Td}(-S)), \end{aligned}$$

the coefficients b_{2k} in the power series expansion (3) are the even Bernoulli numbers. Since

$$\text{Td}(S) - \text{Td}(-S) = \frac{S}{1 - e^{-S}} - \frac{S}{e^S - 1} = S,$$

we have $b_{2k+1} = 0$ for all $k \geq 1$, so the only difference between $\text{Td}(S)$ and $\mathbf{L}(S)$ is the absence of the linear term in $\mathbf{L}(S)$. Replacing $\mathbf{L}(\cdot)$ by $\text{Td}(\cdot)$ on the right hand side of the Euler Maclaurin formula (2) results in a formula for the ordinary sum of the values of f over the integers in $[a, b]$. However, we work with the formula for the weighted sum (1), because this formula avoids “boundary effects” which occur in the formulas for the ordinary sum.

Euler Maclaurin with remainder. As before, we will denote the truncation of the power series $\mathbf{L}(S)$ at the even integer $2k$ by $\mathbf{L}^{2k}(S)$. Then

$$(8) \quad \mathbf{L}^{2k}\left(\frac{\partial}{\partial h}\right) = 1 + \frac{1}{2!}b_2\frac{\partial^2}{\partial h^2} + \dots + \frac{1}{(2k)!}b_{2k}\frac{\partial^{2k}}{\partial h^{2k}}$$

is a differential operator with constant coefficients involving only even order derivatives. In particular, if $g(h)$ is a function with $2k$ continuous derivatives, then

$$(9) \quad \mathbf{L}^{2k}\left(\frac{\partial}{\partial h}\right)(g(h))\Big|_{h=0} = \mathbf{L}^{2k}\left(\frac{\partial}{\partial h}\right)(g(-h))\Big|_{h=0}.$$

The classical Euler Maclaurin summation formula with remainder can be formulated in the following way.

Proposition 10 (Euler Maclaurin with remainder for intervals). *Let $f(x)$ be a function with $m \geq 1$ continuous derivatives and let $k = \lfloor m/2 \rfloor$. Then*

$$(11) \quad \sum'_{[a,b]} f = \mathbf{L}^{2k}\left(\frac{\partial}{\partial h_1}\right)\mathbf{L}^{2k}\left(\frac{\partial}{\partial h_2}\right) \int_{a-h_1}^{b+h_2} f(x)dx \Big|_{h_1=h_2=0} + (-1)^{m-1} \int_a^b P_m(x) f^{(m)}(x)dx$$

with

$$(12) \quad P_m(x) = \frac{B_m(\{x\})}{m!},$$

where $B_m(x)$ is the m th Bernoulli polynomial (see below) and where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x . Moreover, the function $P_m(x)$ is given by

$$(13) \quad P_{2k}(x) = (-1)^{k-1} \sum_{n=1}^{\infty} \frac{2 \cos 2\pi n x}{(2\pi n)^{2k}}$$

if $m = 2k$ is even and by

$$(14) \quad P_{2k+1}(x) = (-1)^{k-1} \sum_{n=1}^{\infty} \frac{2 \sin 2\pi n x}{(2\pi n)^{2k+1}}$$

if $m = 2k + 1$ is odd.

Up to minor changes in notation, this result is formula **298** in [Kn].

Note that if f is a polynomial then (11) becomes an exact formula when m is greater than the degree of f .

Let us recall the proof of Proposition 10. Consider the difference between integration and summation as the difference between two distributions:

$$P_0(x) := 1 - \sum_{k \in \mathbb{Z}} \delta(x - k).$$

Integrating this, with the choice of constant of integration such that the integral from 0 to 1 of the result vanishes, gives a distribution P_1 which is given by the function

$$(15) \quad P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$$

at non-integral points. This is the famous zigzag function studied by Gibbs [Gi]. Notice that P_1 is 1-periodic and odd.

If f is a continuously differentiable function on $[0, 1]$, we get

$$\int_0^1 P_1(x) f'(x) dx = P_1(x) f(x) \Big|_0^1 - \int_0^1 f(x) dx = \frac{1}{2} f(0) + \frac{1}{2} f(1) - \int_0^1 f(x) dx,$$

so

$$\frac{1}{2}f(0) + \frac{1}{2}f(1) = \int_0^1 f(x)dx + \int_0^1 P_1(x)f'(x)dx.$$

Summing the corresponding expressions over the intervals $[a, a+1]$, $[a+1, a+2]$, \dots , $[b-1, b]$, where $a < b$ are integers, gives

$$(16) \quad \sum_{[a,b]}' f = \int_a^b f(x)dx + \int_a^b P_1(x)f'(x)dx.$$

This is the first of a series of expressions relating the sum to an integral.

We now take successive anti-derivatives: for $m \geq 2$, define $P_m(x)$ inductively by the conditions that $\frac{d}{dx}P_m(x) = P_{m-1}(x)$ and $\int_0^1 P_m(x)dx = 0$. Then $P_m(x)$ is 1-periodic and satisfies $P_m(-x) = (-1)^m P_m(x)$. Also, $P_m(x)$ is continuously differentiable $m-2$ times; in particular, it is continuous if $m \geq 2$. Starting from (16), we integrate by parts:

$$\begin{aligned} \sum_{[a,b]}' f &= \int_a^b f(x)dx + P_2 f'|_a^b - \int_a^b P_2(x)f''(x)dx \\ &= \int_a^b f(x)dx + P_2 f'|_a^b - P_3 f''|_a^b + \int_a^b P_3(x)f^{(3)}(x)dx \\ &= \int_a^b f(x)dx + P_2 f'|_a^b - P_3 f''|_a^b + P_4 f^{(3)}|_a^b - \int_a^b P_4(x)f^{(4)}(x)dx \\ &\vdots \end{aligned}$$

Noting that $P_n(a) = P_n(b) = P_n(0)$ and $P_{2n+1}(0) = 0$ (because P_{2n+1} is odd), we get, setting $k = \lfloor m/2 \rfloor$,

$$(17) \quad \sum_{[a,b]}' f = \int_a^b f(x)dx + P_2(0)f'|_a^b + P_4(0)f^{(3)}|_a^b + \dots + P_{2k}(0)f^{(2k-1)}|_a^b \\ + (-1)^{m-1} \int_a^b P_m(x)f^{(m)}(x)dx.$$

Consider the polynomial

$$(18) \quad L^{[2k]}(S) := 1 + P_2(0)S^2 + P_4(0)S^4 + \dots + P_{2k}(0)S^{2k}.$$

From (17) we get the remainder formula

$$(19) \quad \sum_{[a,b]}' f = L^{[2k]}(\frac{\partial}{\partial h_1})L^{[2k]}(\frac{\partial}{\partial h_2}) \int_{a \pm h_1}^{b \pm h_2} f(x)dx \Big|_{h_1=h_2=0} + (-1)^{m-1} \int_a^b P_m(x)f^{(m)}(x)dx$$

for a function f of type C^m , where $k = \lfloor m/2 \rfloor$.

This formula becomes exact when f is a polynomial and when m is sufficiently large. We therefore have, by comparison with Equation (2),

$$(20) \quad L^{[2k]} = \mathbf{L}^{2k}.$$

This and (19) give (11).

It remains to derive the expressions (12), (13), and (14) for the functions $P_m(x)$.

The Bernoulli polynomials $B_m(x)$ are characterized by the properties

$$B_0(x) = 1, \quad \frac{d}{dx}B_m(x) = mB_{m-1}(x), \quad \text{and} \quad \int_0^1 B_m(x)dx = 0 \quad \text{for } m \geq 1.$$

In particular,

$$B_1(x) = x - \frac{1}{2} = P_1(x) \quad \text{for all } 0 < x < 1.$$

(See (15).) Integrating, we get that $P_m(x) = B_m(x)/m!$ for all $0 < x < 1$, and since P_m is 1-periodic, we get (12).

The Poisson summation formula says that, as distributions,

$$\sum_{k \in \mathbb{Z}} \delta(x - k) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x},$$

so

$$P_0(x) = 1 - \sum_{k \in \mathbb{Z}} \delta(x - k) = -2 \sum_{n=1}^{\infty} \cos 2\pi n x,$$

as distributions. Integrating this, with the choice of constant of integration such that the integral from 0 to 1 of the result vanishes, gives the expressions (13) and (14).

The idea of using these Fourier expansions goes back to Wirtinger [W]. See Knopp [Kn], pp. 521–524.

Remark 21. From (18) and (13) we see that the coefficients of the polynomials $L^{[2k]}(S)$ are

$$P_{2k}(0) = (-1)^{k-1} \frac{2}{(2\pi)^{2k}} \sum_{n \geq 1} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{2}{(2\pi)^{2k}} \zeta(2k),$$

where $\zeta(\cdot)$ is the Riemann zeta function. Comparing this with the coefficients of the polynomials $L^{2k}(S)$, we get, from (20) and (8), that

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2} \frac{1}{(2k)!} b_{2k},$$

reproducing Euler’s famous evaluation of the Bernoulli numbers in terms of the Riemann zeta function.

Proposition 10, when applied to a C^m function $f(x)$ of compact support, implies a similar formula for an infinite ray:

$$(22) \quad \frac{1}{2}f(a) + f(a+1) + f(a+2) + \dots \\ = \mathbf{L}^{2k}\left(\frac{\partial}{\partial h}\right) \int_{a \pm h}^{\infty} f(x) dx \Big|_{h=0} + (-1)^{m-1} \int_a^{\infty} P_m(x) f^{(m)}(x) dx$$

for either choice of \pm , where $k = \lfloor m/2 \rfloor$. Indeed, we need only choose b so large that the support of $f(x)$ is contained in the set $\{x < b\}$, and then apply Proposition 10. Conversely, if we know (22) for functions of compact support, then we can conclude (11). Indeed, multiply f by a smooth function of compact support which is equal to one in a neighborhood of $[a, b]$ and observe that

$$(23) \quad \sum'_{[a,b]} = \sum'_{[a,\infty)} - \sum'_{[b,\infty)}$$

when applied to a function of compact support.

Twisted Euler Maclaurin with remainder for a ray. In extending the Euler Maclaurin formula to higher dimensions, we will need an expression for the “twisted weighted sum”

$$\frac{1}{2}f(0) + \sum_{n=1}^{\infty} \lambda^n f(n)$$

when $\lambda \neq 1$ is a root of unity, say, of order N , in terms of the integrals of f .

Let

$$(24) \quad Q_{0,\lambda}(x) = - \sum_{n \in \mathbb{Z}} \lambda^n \delta(x - n).$$

This is an N -periodic distribution since $\lambda^N = 1$. We will take successive anti-derivatives of this distribution, with the constants chosen so that the integrals from 0 to N vanish:

$$(25) \quad \frac{d}{dx} Q_{m,\lambda}(x) = Q_{m-1,\lambda}(x) \quad \text{and} \quad \int_0^N Q_{m,\lambda}(x) dx = 0.$$

With this choice of constants, $Q_{m,\lambda}(x)$ is N -periodic for each m .

Let us now look more closely at $Q_{1,\lambda}(x)$. We have

$$\frac{d}{dx} \mathbf{1}_{[n,n+1]}(x) = \delta(x - n) - \delta(x - (n + 1)),$$

so

$$\frac{d}{dx} \sum_{n \in \mathbb{Z}} \lambda^n \mathbf{1}_{[n,n+1]}(x) = \sum_{n \in \mathbb{Z}} (\lambda^n - \lambda^{n-1}) \delta(x - n) = \frac{1 - \lambda}{\lambda} Q_{0,\lambda}(x).$$

Thus,

$$Q_{1,\lambda}(x) = \frac{\lambda}{1 - \lambda} \sum_{n \in \mathbb{Z}} \lambda^n \mathbf{1}_{[n,n+1]}(x),$$

because the right hand side is periodic of period N and its integral over $[0, N]$ vanishes since $1 + \lambda + \lambda^2 + \dots + \lambda^{N-1} = 0$. If f is a continuously differentiable function of compact support, we have

$$\int_0^\infty Q_{1,\lambda}(x) f'(x) dx = \frac{\lambda}{1 - \lambda} \sum_{n=0}^\infty \lambda^n f \Big|_n^{n+1} = -\frac{\lambda}{1 - \lambda} f(0) + \lambda f(1) + \lambda^2 f(2) + \dots$$

so

$$(26) \quad \frac{1}{2} f(0) + \sum_{n \geq 1} \lambda^n f(n) = \left(\frac{1}{2} + \frac{\lambda}{1 - \lambda} \right) f(0) + \int_0^\infty Q_{1,\lambda}(x) f'(x) dx.$$

Successively applying integration by parts to (26), we get

$$\begin{aligned} & \frac{1}{2} f(0) + \lambda f(1) + \lambda^2 f(2) + \dots \\ &= \left(\frac{1}{2} + \frac{\lambda}{1 - \lambda} \right) f(0) - Q_{2,\lambda}(0) f'(0) - \int_0^\infty Q_{2,\lambda}(x) f''(x) dx \\ &= \left(\frac{1}{2} + \frac{\lambda}{1 - \lambda} \right) f(0) - Q_{2,\lambda}(0) f'(0) + Q_{3,\lambda}(0) f''(0) + \int_0^\infty Q_{3,\lambda}(x) f^{(3)}(x) dx \\ & \quad \vdots \\ &= \left(\frac{1}{2} + \frac{\lambda}{1 - \lambda} \right) f(0) - Q_{2,\lambda}(0) f'(0) + \dots + (-1)^{k-1} Q_{k,\lambda}(0) f^{(k-1)}(0) \\ & \quad + (-1)^{k-1} \int_0^\infty Q_{k,\lambda}(x) f^{(k)}(x) dx. \end{aligned}$$

Since

$$(-1)^{m-1} f^{(m-1)}(0) = \left(\frac{\partial}{\partial h} \right)^m \int_{-h}^\infty f(x) dx \Big|_{h=0},$$

we have thus proved this “twisted Euler Maclaurin formula for a ray”:

Proposition 27. *Let*

$$\mathbf{M}^{k,\lambda}(S) = \left(\frac{1}{2} + \frac{\lambda}{1-\lambda}\right) S + Q_{2,\lambda}(0)S^2 + Q_{3,\lambda}(0)S^3 + \cdots + Q_{k,\lambda}(0)S^k,$$

for a root of unity $\lambda \neq 1$. Then

$$(28) \quad \frac{1}{2}f(0) + \lambda f(1) + \lambda^2 f(2) + \cdots = \mathbf{M}^{k,\lambda}\left(\frac{\partial}{\partial h}\right) \int_{-h}^{\infty} f(x)dx \Big|_{h=0} + (-1)^{k-1} \int_0^{\infty} Q_{k,\lambda}(x) f^{(k)}(x) dx$$

if $f \in C_c^k(\mathbb{R})$.

Remark 29. Another twisted Euler Maclaurin formula for a ray appeared in [T].

We will now show that the polynomials that appear in Proposition 27 have the following symmetry property:

$$(30) \quad \mathbf{M}^{m,\lambda^{-1}}(S) = \mathbf{M}^{m,\lambda}(-S).$$

The linear term transforms according to (30) because

$$\frac{1}{2} + \frac{\lambda^{-1}}{1-\lambda^{-1}} = \frac{1}{2} - \left(1 + \frac{\lambda}{1-\lambda}\right) = -\left(\frac{1}{2} + \frac{\lambda}{1-\lambda}\right).$$

For the other terms to transform correctly, we need to check that

$$(31) \quad Q_{m,\lambda^{-1}}(0) = (-1)^m Q_{m,\lambda}(0) \quad \text{for all } m \geq 2.$$

As in the non-twisted formula, it is convenient to work with the Fourier expansions of the $Q_{m,\lambda}$'s. Let

$$\lambda = e^{2\pi i \frac{j}{N}}.$$

We are assuming that $\lambda \neq 1$, and so $j \not\equiv 0 \pmod{N}$. Then we can write the distribution $Q_{0,\lambda}(x)$ as

$$Q_{0,\lambda}(x) = - \sum_{n \in \mathbb{Z}} \lambda^n \delta(x-n) = -e^{(2\pi i \frac{j}{N})x} \sum_{n \in \mathbb{Z}} \delta(x-n).$$

Writing

$$\sum_{n \in \mathbb{Z}} \delta(x-n) = \sum_{r \in \mathbb{Z}} e^{2\pi i r x},$$

we see that the Fourier series of $Q_{0,\lambda}(x)$ is

$$Q_{0,\lambda}(x) = - \sum_{r \in \mathbb{Z}} e^{2\pi i (\frac{j}{N} + r)x}.$$

The indefinite integral, chosen so that the integral over $[0, N]$ vanishes, is obtained by dividing each Fourier summand by the coefficient of the exponent, so we have the following Fourier series:

$$(32) \quad Q_{m,\lambda}(x) = - \sum_{r \in \mathbb{Z}} \frac{e^{2\pi i (\frac{j}{N} + r)x}}{\left(2\pi i (\frac{j}{N} + r)\right)^m}.$$

Setting $x = 0$, replacing j by $-j$, and replacing r by $-r$ in the sum, pulls out a factor of $(-1)^m$, which gives the required equation (31), and which implies (30).

Remark 33. For $\lambda = 1$, if we define

$$\mathbf{M}^{k,1}(S) = \mathbf{L}^{2\lfloor k/2 \rfloor}(S) \quad \text{and} \quad Q_{k,1} = P_k,$$

then (28) boils down to (22). So, with this notation, (28) also holds for $\lambda = 1$. Notice that if $\lambda \neq 1$ then $M^{k,\lambda}(S)$ is a multiple of S , and that if $\lambda = 1$ then $M^{k,\lambda}(S) = 1 +$ a multiple of S . Finally, the symmetry property (30) continues to hold for $\lambda = 1$ because the polynomials \mathbf{L}^{2k} are symmetric.

3. THE POLAR DECOMPOSITION OF A SIMPLE POLYTOPE

In this section we decompose a polytope into an “alternating sum of polyhedral cones”. See [V] or [L]. We will give a “weighted version” of this decomposition.

Our polytopes are always compact and convex. A compact convex polytope Δ in \mathbb{R}^n is a compact set which can be obtained as the intersection of finitely many half-spaces, say,

$$(34) \quad \Delta = H_1 \cap \dots \cap H_d.$$

We assume that (34) is an intersection with the smallest possible d , so that the H_i ’s are uniquely determined up to permutation. We order them arbitrarily. The *facets* (codimension one faces) of Δ are

$$\sigma_i = \Delta \cap \partial H_i \quad , \quad i = 1, \dots, d.$$

Alternatively, a compact convex polytope is the convex hull of a finite set of points in \mathbb{R}^n . If we take this set to be minimal, it is uniquely determined, and its elements are the *vertices* of Δ .

For each vertex v of Δ , let $I_v \subset \{1, \dots, d\}$ encode the set of facets that contain v , so that

$$i \in I_v \quad \text{if and only if} \quad v \in \sigma_i.$$

Assume that Δ is *simple*, so that each vertex is the intersection of exactly n facets. For each $i \in I_v$, there exists a unique edge at v which does not belong to the facet σ_i ; choose any vector $\alpha_{i,v}$ in the direction of this edge. (At the moment, these “edge vectors” are only determined up to a positive scalar. Later, when discussing integral polytopes, we will make a specific choice of the edge vectors.)

The *tangent cone* to Δ at v is

$$\mathbf{C}_v = \{v + r(x - v) \mid r \geq 0, x \in \Delta\} = v + \sum_{j \in I_v} \mathbb{R}_{\geq 0} \alpha_{j,v}.$$

We will “polarize” these tangent cones by flipping some of their edges so that they all “point in the same direction”. This direction is specified by the choice of a “polarizing vector”: a vector $\xi \in \mathbb{R}^{n*}$, such that $\langle \xi, \alpha_{j,v} \rangle$ is non-zero for all v and j . With this choice, we define the *polarized edge vectors* to be

$$(35) \quad \alpha_{i,v}^\sharp = \begin{cases} \alpha_{i,v} & \text{if } \langle \xi, \alpha_{i,v} \rangle < 0, \\ -\alpha_{i,v} & \text{if } \langle \xi, \alpha_{i,v} \rangle > 0, \end{cases}$$

and the *polarized tangent cone* to be

$$(36) \quad \mathbf{C}_v^\sharp = v + \sum_{j \in I_v} \mathbb{R}_{\geq 0} \alpha_{j,v}^\sharp.$$

We define the “weighted characteristic function”,

$$(37) \quad \mathbf{1}_\Delta^w(x),$$

to be the function on \mathbb{R}^n that takes the value 0 on the exterior of Δ , the value 1 on the interior of Δ , and the value $1/2^k$ on the relative interior of a codimension k face of Δ . So, for example, for an interval $[a, b]$ on the line, the function $\mathbf{1}_{[a,b]}^w(x)$ assigns the value one to points $a < x < b$, zero to points outside the interval, and $\frac{1}{2}$ to the points a and b . We use a similar definition for a polyhedral cone.

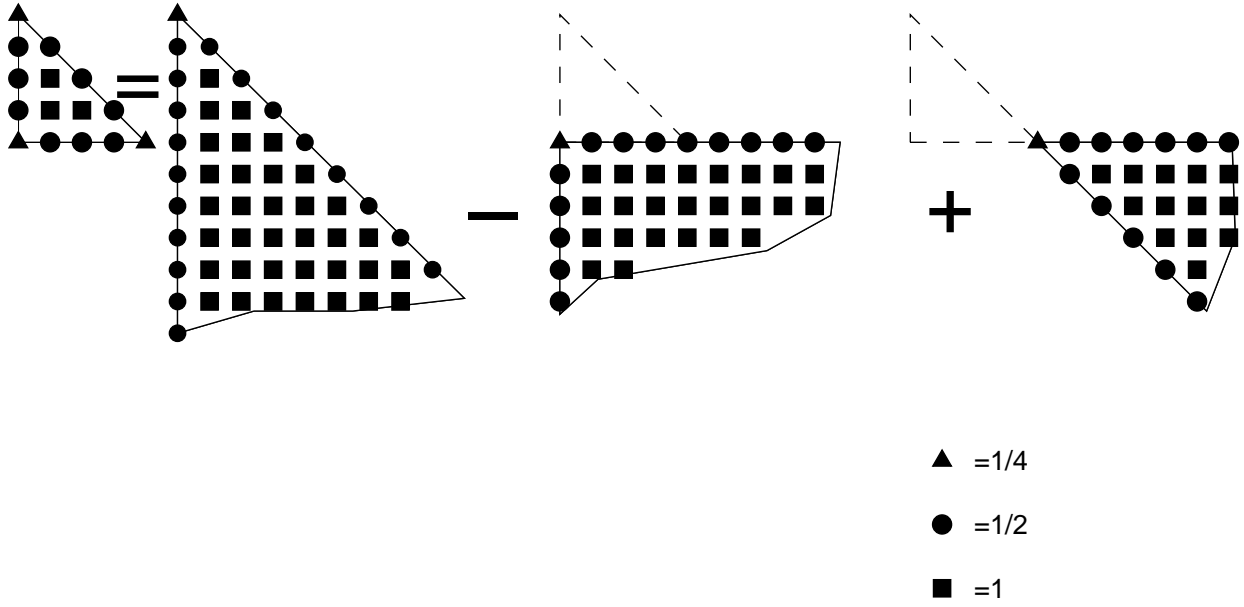


FIGURE 1. Polar decomposition of a triangle

Proposition 38 (Weighted polar decomposition of a simple polytope). *Let Δ be a simple polytope. For any choice of polarizing vector, we have*

$$(39) \quad \mathbf{1}_{\Delta}^w(x) = \sum_v (-1)^{\#v} \mathbf{1}_{\mathbf{C}_v^{\sharp}}^w(x),$$

where the sum is over the vertices v of Δ , where \mathbf{C}_v^{\sharp} is the polarized tangent cone, where $\mathbf{1}_{\Delta}^w(x)$ and $\mathbf{1}_{\mathbf{C}_v^{\sharp}}^w(x)$ are the weighted characteristic functions, and where $\#v$ denotes the number of edge vectors at v whose signs were changed by the polarizing process (35).

This theorem is illustrated for the case of a triangle in Figure 1.

Proof. We will prove this equality in two steps. First, for each x we will find a polarization such that the equality (39) holds. Second, we will show that the right hand side of the equality (39) is independent of the choice of polarization.

Suppose that $x \notin \Delta$. Let ξ be a polarizing vector such that $\langle \xi, x \rangle > \langle \xi, y \rangle$ for all $y \in \Delta$. Then all the polarized cones “point away from x ” and do not contain x . Formula (39) for the polarizing vector ξ , when evaluated at x , states that $0 = 0$.

For every polarization there exists exactly one vertex v for which $\mathbf{C}_v^{\sharp} = \mathbf{C}_v$, namely, the vertex v such that $\langle \xi, v \rangle$ is maximal. Conversely, for every vertex v there exists a polarization such that $\mathbf{C}_v^{\sharp} = \mathbf{C}_v$.

Suppose that $x \in \Delta$ is in the relative interior of a face F of codimension k . Let v be any vertex of F . Let ξ be a polarizing vector such that $\mathbf{C}_v^{\sharp} = \mathbf{C}_v$. Let F_v be the codimension k face of \mathbf{C}_v that contains F . Then $\mathbf{1}_{\Delta}^w(x) = \mathbf{1}_{\mathbf{C}_v^{\sharp}}^w(x) = (\frac{1}{2})^k$. For each other vertex, v' , the cone $\mathbf{C}_{v'}^{\sharp}$ is disjoint from the relative interior of F , so $\mathbf{1}_{\mathbf{C}_{v'}^{\sharp}}^w(x) = 0$. Formula (39) for this polarization, when evaluated at x , states that $(\frac{1}{2})^k = (\frac{1}{2})^k$.

To keep track of the different possible choices of polarization, we let

$$E_1, \dots, E_N$$

denote all the different codimension one subspaces of \mathbb{R}^{n*} that are equal to

$$\alpha_{j,v}^\perp = \{\eta \in \mathbb{R}^{n*} \mid \langle \eta, \alpha_{j,v} \rangle = 0\}$$

for some j and v . (For instance, if no two edges of Δ are parallel, then the number N of such hyperplanes is equal to the number of edges of Δ .) A vector ξ can be taken to be a “polarizing vector” if and only if it does not belong to any E_j . The “polarized cones” \mathbf{C}_v^\sharp only depend on the connected component of the complement

$$(40) \quad \mathbb{R}^{n*} \setminus (E_1 \cup \dots \cup E_N)$$

in which ξ lies. Any two polarizing vectors can be connected by a path ξ_t in \mathbb{R}^{n*} which crosses the “walls” E_j one at a time. We finish by showing that the right hand side of formula (39) does not change when the polarizing vector ξ_t crosses a single wall, E_k .

As ξ_t crosses the wall E_k , the sign of the pairing $\langle \xi_t, \alpha_{j,v} \rangle$ flips exactly if $E_k = \alpha_{j,v}^\perp$. For each vertex v , denote by $\mathbf{S}_v(x)$ and $\mathbf{S}'_v(x)$ its contributions to the right hand side of formula (39) before and after ξ_t crossed the wall. The vertices for which these contributions differ are exactly those vertices that lie on edges e of Δ which are perpendicular to E_k . They come in pairs because each edge has two endpoints.

Let us concentrate on one such an edge, e , with endpoints, say, u and v . Let α_e denote an edge vector at v that points from v to u along e . Suppose that the pairing $\langle \xi_t, \alpha_e \rangle$ flips its sign from negative to positive as ξ_t crosses the wall; (otherwise we switch the roles of v and u). The “polarized tangent cones” to Δ at v , before and after ξ_t crosses the wall, are

$$\begin{aligned} \mathbf{C}_v^\sharp &= v + \sum_{j \in I_e} \mathbb{R}_{\geq 0} \alpha_{j,v}^\sharp + \mathbb{R}_{\geq 0} \alpha_e \quad \text{and} \\ (\mathbf{C}_v^\sharp)' &= v + \sum_{j \in I_e} \mathbb{R}_{\geq 0} \alpha_{j,v}^\sharp - \mathbb{R}_{\geq 0} \alpha_e \end{aligned}$$

where $I_e \subset \{1, \dots, d\}$ encodes the facets that contain e . (The $\alpha_{j,v}^\sharp$ are the same for the different ξ_t 's because the pairings $\langle \xi_t, \alpha_{j,v}^\sharp \rangle$ do not flip sign when ξ_t crosses the wall for $j \in I_e$.) The cones \mathbf{C}_v^\sharp and $(\mathbf{C}_v^\sharp)'$ have a common facet and their union is

$$\mathbf{C}_e^\sharp := v + \sum_{j \in I_e} \mathbb{R}_{\geq 0} \alpha_{j,v}^\sharp + \mathbb{R} \alpha_e.$$

This union only depends on the edge e and not on the endpoint v . (This uses the assumption that the polytope Δ is simple and follows from the fact that $\alpha_{j,u} \in \mathbb{R}_{\geq 0} \alpha_{j,v} + \mathbb{R} \alpha_e$.)

The contributions of v to the right hand side of (39) before and after ξ_t crosses the wall are

$$(41) \quad \mathbf{S}_v(x) = \varepsilon \mathbf{1}_{\mathbf{C}_v^\sharp}^w \quad \text{and} \quad \mathbf{S}'_v(x) = -\varepsilon \mathbf{1}_{(\mathbf{C}_v^\sharp)'}^w$$

where $\varepsilon \in \{-1, 1\}$. Their difference is plus/minus the weighted characteristic function of \mathbf{C}_e^\sharp :

$$(42) \quad \mathbf{S}_v(x) - \mathbf{S}'_v(x) = \varepsilon \mathbf{1}_{\mathbf{C}_e^\sharp}^w$$

The contributions of the other endpoint, u , have opposite signs than the respective contributions (41) of v , and their difference is minus/plus the characteristic function of \mathbf{C}_e^\sharp . Hence, the differences $\mathbf{S}_v(x) - \mathbf{S}'_v(x)$ and $\mathbf{S}_u(x) - \mathbf{S}'_u(x)$, for the two endpoints u and v of e , sum to zero. \square

4. EULER MACLAURIN FORMULA WITH REMAINDER FOR REGULAR POLYTOPES.

We extend our notation (1) to a weighted sum over a simple integral polytope Δ :

$$\sum_{\Delta \cap \mathbb{Z}^n} 'f := \sum_{x \in \Delta \cap \mathbb{Z}^n} \mathbf{1}_\Delta^w(x) f(x),$$

where $\mathbf{1}_\Delta^w(x)$ is the weighted characteristic function, introduced in (37), which is equal to $1/2^k$ when x lies in the relative interior of a face of Δ of codimension k . We use similar notation for the weighted sum of a compactly supported function f over a simple polyhedral cone.

With this notation, Proposition 38 gives the following decomposition. Let $\xi \in \mathbb{R}^{n*}$ be a “polarizing vector”. Then

$$(43) \quad \sum_{\Delta \cap \mathbb{Z}^n} 'f = \sum_v (-1)^{\#v} \sum_{\mathbf{C}_v^\# \cap \mathbb{Z}^n} 'f$$

where we sum over the vertices v of Δ , where $\mathbf{C}_v^\#$ is the polarized tangent cone to Δ at v (see (36)), and where $\#v$ is the number of edge vectors at v that are flipped by the polarization process (35).

For the standard closed orthant $\mathbf{O} = \prod_{i=1}^n \mathbb{R}_{\geq 0}$ in \mathbb{R}^n , we have

$$\sum_{\mathbf{O} \cap \mathbb{Z}^n} 'g = \sum_{m_1 \in \mathbb{Z}_{\geq 0}} ' \cdots \sum_{m_n \in \mathbb{Z}_{\geq 0}} 'g(m_1, \dots, m_n),$$

for any function g of compact support.

By performing n iterations of (22), we obtain an Euler Maclaurin formula with remainder for the standard orthant: Let m be an integer and let $k = \lfloor m/2 \rfloor$. If g is a C^{mn} function of compact support and $m \geq 1$, then for any choice of the \pm 's we have

$$(44) \quad \sum_{\mathbf{O} \cap \mathbb{Z}^n} 'g = \prod_{i=1}^n \mathbf{L}^{2k} \left(\frac{\partial}{\partial h_i} \right) \int_{\mathbf{O}(\pm h_1, \dots, \pm h_n)} g(x) dx \Big|_{h_1 = \dots = h_n = 0} + R_m^{\text{st}}(g),$$

where

$$\mathbf{O}(h_1, \dots, h_n) = \{t \mid t_i + h_i \geq 0 \ \forall i\}$$

denotes the shifted orthant, and where the remainder term is given by

$$(45) \quad R_m^{\text{st}}(g) = \sum_{I \subsetneq \{1, \dots, n\}} (-1)^{(m-1)(n-|I|)} \prod_{i \in I} \mathbf{L}^{2k} \left(\frac{\partial}{\partial h_i} \right) \int_{\mathbf{O}(\pm h_1, \dots, \pm h_n)} \prod_{i \notin I} P_m(x_i) \prod_{i \notin I} \left(\frac{\partial}{\partial x_i} \right)^m g(x) dx_1 \cdots dx_n \Big|_{h=0}.$$

This remainder can also be expressed as a sum of integrals over the orthant of bounded periodic functions times various partial derivatives of f of order no less than m and no more than mn . This fact follows from the formula

$$(46) \quad \frac{\partial}{\partial h_i} \int_{\mathbf{O}(h_1, \dots, h_n)} \varphi dx_1 \cdots dx_n = - \int_{\mathbf{O}(h_1, \dots, h_n)} \frac{\partial \varphi}{\partial x_i} dx_1 \cdots dx_n.$$

We can apply iterations of this formula with $i \in I$ to the I 'th summand of (45) because the non-smooth functions P_m are only applied to the variables x_j for $j \notin I$. We get

$$(47) \quad R_m^{\text{st}}(g) = \sum_{I \subsetneq \{1, \dots, n\}} (-1)^{(m-1)(n-|I|)} \int_{\mathbf{O}} \prod_{i \in I} \mathbf{L}^{2k} \left(-\frac{\partial}{\partial x_i} \right) \prod_{i \notin I} P_m(x_i) \prod_{i \notin I} \left(\frac{\partial}{\partial x_i} \right)^m g(x) dx_1 \cdots dx_n.$$

A regular integral orthant \mathbf{C} is the image of the standard orthant \mathbf{O} via an affine transformation of the form

$$(48) \quad (t_1, \dots, t_n) \mapsto x = v + t_1 \alpha_1 + \dots + t_n \alpha_n$$

where $\alpha_1, \dots, \alpha_n$ generate \mathbb{Z}^n and where $v \in \mathbb{Z}^n$. If $u_1, \dots, u_n \in \mathbb{R}^{n*}$ is the dual basis to $\alpha_1, \dots, \alpha_n$ then the image of $\mathbf{O}(h_1, \dots, h_n)$ under this transformation is given by the inequalities

$$(49) \quad \langle u_i, x \rangle - \langle u_i, v \rangle + h_i \geq 0.$$

We denote this expanded orthant by $\mathbf{C}(h)$. If f is a C^{mn} function of compact support and

$$g(t_1, \dots, t_n) = f(t_1\alpha_1 + \dots + t_n\alpha_n)$$

is its pullback under the transformation (48), then

$$\sum_{\mathbf{C} \cap \mathbb{Z}^n} 'f = \sum_{\mathbf{O} \cap \mathbb{Z}^n} 'g \quad \text{and} \quad \int_{\mathbf{C}(h)} f(x) dx = \int_{\mathbf{O}(h)} g(t) dt,$$

and so we have an Euler Maclaurin formula for regular orthants:

$$(50) \quad \sum_{\mathbf{C} \cap \mathbb{Z}^n} 'f = \prod_{i=1}^n \mathbf{L}^{2k} \left(\frac{\partial}{\partial h_i} \right) \int_{\mathbf{C}(\pm h_1, \dots, \pm h_n)} f(x) dx \Big|_{h_1=\dots=h_n=0} + R_m^{\mathbf{C}}(f),$$

where

$$R_m^{\mathbf{C}}(f) = R_m^{\text{st}}(g).$$

Let Δ be a compact convex polytope. As in Section 3, Δ can be written as an intersection of half-spaces

$$(51) \quad \Delta = H_1 \cap \dots \cap H_d, \quad \text{where} \quad H_i = \{x \mid \langle u_i, x \rangle + \mu_i \geq 0\} \quad \text{for } i = 1, \dots, d$$

and d is the number of facets of Δ . The vector $u_i \in \mathbb{R}^{n*}$ can be thought of as the inward normal to the i th facet of Δ ; a-priori it is determined up to multiplication by a positive number. If all the vertices of Δ are integral, then the u_i 's can be chosen to belong to the dual lattice \mathbb{Z}^{n*} , and we can fix our choice of the u_i 's by imposing the normalization condition that the u_i 's be primitive lattice elements, that is, that no u_i can be expressed as a multiple of a lattice element by an integer greater than one. (The fact that a normal vector u to a facet σ can be chosen to be integral is a consequence of Cramer's rule. Indeed, we can choose integral edge vectors β_1, \dots, β_n that emanate from a vertex on σ such that $\beta_1, \dots, \beta_{n-1}$ span the tangent plane to σ and β_n is transverse to σ . Solving the linear equations $\langle u, \beta_1 \rangle = \dots = \langle u, \beta_{n-1} \rangle = 0$ and $\langle u, \beta_n \rangle = 1$, we get an inward normal vector u with rational entries; clearing denominators, we may assume that u is actually integral.)

We can then consider the “dilated polytope” $\Delta(h_1, \dots, h_d)$, which is obtained by shifting the i th facet outward by a “distance” h_i . More precisely,

$$\Delta(h) = \bigcap_{i=1}^d \{x \mid \langle u_i, x \rangle + \mu_i + h_i \geq 0\} \quad \text{where} \quad h = (h_1, \dots, h_d).$$

Now assume that Δ is simple. Then $\Delta(h)$ is simple if h is sufficiently small. The polar decomposition of $\Delta(h)$ involves “dilated orthants”. However, dilating the facets of Δ outward results in dilating some facets of \mathbf{C}_v^\sharp inward and some outward. Explicitly, for $i \in I_v = \{i_1, \dots, i_n\}$, the inward normal vector to the i th facet of \mathbf{C}_v^\sharp is

$$(52) \quad u_{i,v}^\sharp = \begin{cases} u_i & \text{if } \alpha_{i,v}^\sharp = \alpha_{i,v} \\ -u_i & \text{if } \alpha_{i,v}^\sharp = -\alpha_{i,v}. \end{cases}$$

Hence, the dilated orthants that occur on the right hand side of the polar decomposition of $\Delta(h)$ are $\mathbf{C}_v^\sharp(h_{i_1,v}^\sharp, \dots, h_{i_n,v}^\sharp)$, where

$$(53) \quad h_{i,v}^\sharp = \begin{cases} h_i & \text{if } \alpha_{i,v}^\sharp = \alpha_{i,v} \\ -h_i & \text{if } \alpha_{i,v}^\sharp = -\alpha_{i,v}. \end{cases}$$

This subtlety in the signs does not effect the formula for regular polytopes, because of the symmetry of \mathbf{L}^{2k} ; however, it is needed to derive the formula for simple polytopes.

If Δ is a regular integral polytope, then the \mathbf{C}_v^\sharp 's are regular integral orthants, and the “dilated orthants” are exactly as in (49). We then have, by (43), (50), and the symmetry of \mathbf{L}^{2k} ,

$$(54) \quad \sum_{\Delta \cap \mathbb{Z}^n} 'f = \sum_v (-1)^{\#v} \sum_{\mathbf{C}_v^\sharp \cap \mathbb{Z}^n} 'f \\ = \sum_v (-1)^{\#v} \left(\prod_{i \in I_v = \{i_1, \dots, i_n\}} \mathbf{L}^{2k}(\frac{\partial}{\partial h_i}) \int_{\mathbf{C}_v^\sharp(\pm h_{i_1}, \dots, \pm h_{i_n})} f(x) dx \right) \Big|_{h=0} + R_m^{\mathbf{C}_v^\sharp}(f) \Big|_{h=0}.$$

We may multiply the differential operator $\prod \mathbf{L}^{2k}(\frac{\partial}{\partial h_i})$ in the above expression by any number of operators of the form $\mathbf{L}^{2k}(\frac{\partial}{\partial h_j})$ where $j \notin I_v$, since all that will remain of these operators is the constant term 1, all actual differentiations yielding zero. The right hand side of (54) is then equal to

$$\prod_{i=1}^d \mathbf{L}^{2k}(\frac{\partial}{\partial h_i}) \sum_v (-1)^{\#v} \int_{\mathbf{C}_v^\sharp(h_{i_1,v}^\sharp, \dots, h_{i_n,v}^\sharp)} f(x) dx \Big|_{h=0} + R_m^{\mathbf{C}_v^\sharp}(f) \\ = \prod_{i=1}^d \mathbf{L}^{2k}(\frac{\partial}{\partial h_i}) \int_{\Delta(h_1, \dots, h_d)} f(x) dx \Big|_{h=0} + S_\Delta^m(f)$$

where

$$(55) \quad S_\Delta^m(f) := \sum_v (-1)^{\#v} R_m^{\mathbf{C}_v^\sharp}(f)$$

and where $\{i_1, \dots, i_n\} = I_v$ in the v 'th summand.

Notice that both

$$\sum_{\Delta \cap \mathbb{Z}^n} 'f$$

and

$$\prod_{i=1}^d \mathbf{L}^{2k}(\frac{\partial}{\partial h_i}) \int_{\Delta(h_1, \dots, h_d)} f(x) dx \Big|_{h=0}$$

do not depend on the choice of polarization, and both vanish on any function f whose support is disjoint from the polytope. So the same must be true of the remainder. We have proved the following result:

Theorem 1 ([KSW]). *Let $m \geq 1$ be an integer. Let $\Delta \subset \mathbb{R}^n$ be a regular integral polytope and f a C^{mn} function of compact support on \mathbb{R}^n . Choose a “polarizing vector” for Δ . Then*

$$\sum_{\Delta \cap \mathbb{Z}^n} ' f = \prod_{i=1}^d \mathbf{L}^{2k} \left(\frac{\partial}{\partial h_i} \right) \int_{\Delta(h_1, \dots, h_d)} f(x) dx \Big|_{h=0} + S_{\Delta}^m(f)$$

where $k = \lfloor m/2 \rfloor$ and where $S_{\Delta}^m(f)$ is given by (55). This remainder can be expressed as a sum of integrals over orthants of bounded periodic functions times various partial derivatives of f of order no less than m and no more than mn . Finally, this remainder is independent of the choice of polarization and is a distribution supported on the polytope Δ .

5. FINITE GROUPS ASSOCIATED TO A SIMPLE INTEGRAL POLYTOPE AND ITS FACES.

In order to extend Theorem 1 to simple integral polytopes that may not be regular, we must extend the Euler Maclaurin formula for regular orthants (50) to a formula that is valid for simple orthants that may not be regular. In this section we analyze certain finite groups that arise in this generalization.

Let \mathbf{C} be a simple integral orthant. This means that we can write \mathbf{C} as the intersection of n half-planes in general position,

$$(56) \quad \mathbf{C} = H_1 \cap \dots \cap H_n \quad \text{where} \quad H_i = \{x \mid \langle u_i, x \rangle + \mu_i \geq 0\} \quad \text{for } i = 1, \dots, n,$$

and that the vertex of \mathbf{C} is in \mathbb{Z}^n . The u_i 's are inward normals to the facets of \mathbf{C} , and we choose them to be primitive elements of the dual lattice \mathbb{Z}^{n*} . (See Section 4.) We choose $\alpha_1, \dots, \alpha_n$ to be the dual basis, that is,

$$\langle u_j, \alpha_i \rangle = \begin{cases} 1 & j = i \\ 0 & j \neq i. \end{cases}$$

The α_i 's are edge vectors of \mathbf{C} , but they might not be integral: they generate a lattice in \mathbb{R}^n which is a finite extension of \mathbb{Z}^n . This extension is trivial exactly if Δ is regular at v , that is, if the u_i 's, generate the dual lattice \mathbb{Z}^{n*} . To the cone \mathbf{C} we associate the finite group

$$(57) \quad \Gamma := \mathbb{Z}^{n*} / \sum \mathbb{Z} u_i.$$

So this group is trivial exactly if the cone \mathbf{C} is regular.

Lemma 58. *In the above setting,*

$$(59) \quad \gamma \mapsto e^{2\pi i \langle \gamma, x \rangle}$$

is a well defined character on Γ whenever $x = m_1 \alpha_1 + \dots + m_n \alpha_n$ where m_j are integers. This character is trivial if and only if $x \in \mathbb{Z}^n$.

Proof. Let $\tilde{\gamma} \in \mathbb{Z}^{n*}$ be an element that represents γ . If we expand it as a combination of the basis elements u_j , so that

$$\tilde{\gamma} = b_1 u_1 + \dots + b_n u_n,$$

then $\langle \tilde{\gamma}, \alpha_j \rangle = b_j$ for all j . If $\tilde{\gamma}'$ is another element of \mathbb{Z}^{n*} that represents γ , then it differs from $\tilde{\gamma}$ by an integral combination of the u_i 's. So $\langle \tilde{\gamma}', \alpha_j \rangle$ differs from $\langle \tilde{\gamma}, \alpha_j \rangle$ by an integer. Hence, when x is an integral combination of the α_j 's, the pairing $\langle \tilde{\gamma}, x \rangle$ is well defined modulo \mathbb{Z} , and so $e^{2\pi i \langle \gamma, x \rangle}$ is well defined.

Finally, the character (59) is trivial if and only if $\langle \tilde{\gamma}, x \rangle$ is an integer for all $\gamma \in \mathbb{Z}^{n*}$, and this holds if and only if $x \in \mathbb{Z}^n$. \square

Let Δ be a simple integral polytope in \mathbb{R}^n , given by (51). For each vertex v of Δ , let $I_v \subset \{1, \dots, d\}$ denote the set of facets of Δ that meet at v . The normal vectors u_i , $i \in I_v$, form a basis of \mathbb{R}^{n*} . We choose

$$(60) \quad \alpha_{i,v} \in \mathbb{R}^n \quad , \quad i \in I_v$$

to be the dual basis.

Given any face F of the polytope Δ , let I_F denote the set of facets of Δ which meet at F . Because Δ is simple, the vectors u_i , for $i \in I_F$, are linearly independent. Let $N_F \subseteq \mathbb{R}^{n*}$ be the subspace

$$N_F = \text{span}\{u_i \mid i \in I_F\}.$$

Remark 61. It is natural to define the tangent space to the face F to be $TF = \text{span}\{x-y \mid x, y \in F\}$, the normal space to F to be the quotient \mathbb{R}^n/TF , and the co-normal space to be the dual of the normal space. With these definitions, N_F is the co-normal space to F .

To each face F of Δ we associate a finite abelian group Γ_F . Explicitly, the lattice

$$V_F = \sum_{i \in I_F} \mathbb{Z}u_i \subset N_F$$

is a sublattice of $N_F \cap \mathbb{Z}^{n*}$ of finite index, and the finite abelian group associated to the face F is the quotient

$$(62) \quad \Gamma_F := (N_F \cap \mathbb{Z}^{n*})/V_F.$$

If $F = v$ is a vertex, this is the same as the finite abelian group associated to the tangent cone \mathbf{C}_v as in (57).

Let E and F be two faces of Δ with $F \subseteq E$. This inclusion implies that $I_E \subseteq I_F$, and hence

$$(63) \quad \{u_i\}_{i \in I_E} \subseteq \{u_i\}_{i \in I_F}.$$

Because these sets are bases of the vector spaces N_E and N_F , we have an inclusion

$$N_E \subseteq N_F.$$

Because the sets occurring in (63) are \mathbb{Z} -bases of the lattices V_E and V_F , we have $N_E \cap V_F = V_E$. Hence, the natural map from $\Gamma_E = (\mathbb{Z}^{n*} \cap N_E)/V_E$ to $\Gamma_F = (\mathbb{Z}^{n*} \cap N_F)/V_F$ is one to one, and it provides us with a natural inclusion map:

$$\text{if } F \subseteq E \text{ then } \Gamma_E \subseteq \Gamma_F.$$

We define a subset Γ_F^\flat of Γ_F by

$$(64) \quad \Gamma_F^\flat := \Gamma_F \setminus \bigcup_{\text{faces } E \text{ such that } E \supsetneq F} \Gamma_E.$$

Then

$$(65) \quad \Gamma_v = \bigsqcup_{\{F: v \in F\}} \Gamma_F^\flat.$$

Recall that

$$(66) \quad \lambda_{\gamma,j,v} := e^{2\pi i \langle \gamma, \alpha_{j,v} \rangle}, \quad \text{for } \gamma \in \Gamma_v \quad \text{and } j \in I_v,$$

is well defined, by Lemma 58. It is a root of unity. We will need the following results.

Claim 67. *If $\gamma \in \Gamma_F$ and $j \in I_F$, then $\lambda_{\gamma,j,v}$ is the same for all $v \in F$.*

This allows us to define $\lambda_{\gamma,j,F}$ for $\gamma \in \Gamma_F$ and $j \in I_F$ such that

$$\lambda_{\gamma,j,F} = \lambda_{\gamma,j,v} \quad \text{for } \gamma \in \Gamma_F \text{ and } j \in I_F, \text{ if } v \in F.$$

Claim 68. *If $\gamma \in \Gamma_F$ and $j \in I_v \setminus I_F$ then $\lambda_{\gamma,j,v}$ is equal to one.*

This allows us to define $\lambda_{\gamma,j,F} = 1$ when $\gamma \in \Gamma_F$ and $j \in \{1, \dots, d\} \setminus I_F$. Then

$$(69) \quad \lambda_{\gamma,j,F} = \lambda_{\gamma,j,v} \quad \text{for } \gamma \in \Gamma_F \text{ and } 1 \leq j \leq d, \text{ if } v \in F$$

and

$$(70) \quad \lambda_{\gamma,j,F} = 1 \quad \text{for } \gamma \in \Gamma_F \text{ if } j \notin I_F.$$

Claim 71. *If $\gamma \in \Gamma_F^\flat$ and $j \in I_F$, then $\lambda_{\gamma,j,F} \neq 1$.*

Proof of Claims 67, 68, and 71. Let $\gamma \in \Gamma_F$ be represented by

$$\tilde{\gamma} = \sum_{i \in I_F} b_i u_i \in N_F \cap \mathbb{Z}^{n*}$$

for some $b_i \in \mathbb{R}$. (See (62).)

Let $v \in F$. Because $\alpha_{j,v}$, $j \in I_v$, is a dual basis to u_j , $j \in I_v$, we have

$$\langle \tilde{\gamma}, \alpha_{j,v} \rangle = \begin{cases} b_j & j \in I_F \\ 0 & j \in I_v \setminus I_F. \end{cases}$$

Hence,

$$\lambda_{\gamma,j,v} = e^{2\pi i \langle \tilde{\gamma}, \alpha_{j,v} \rangle} = \begin{cases} e^{2\pi i b_j} & j \in I_F \\ 1 & j \in I_v \setminus I_F \end{cases}$$

is independent of v and is equal to 1 if $j \in I_v \setminus I_F$. This prove Claims 67 and 68.

Let $j \in I_F$. If $\lambda_{\gamma,j,F} := e^{2\pi i b_j}$ is equal to one, then b_j is an integer, so

$$(72) \quad \tilde{\gamma}' = \sum_{i \in I_F \setminus \{j\}} b_i u_i$$

also represents γ . Let $E \supset F$ be the face such that $I_E = I_F \setminus \{j\}$. Then, by (72), $\gamma \in \Gamma_E$. This proves Claim 71. \square

6. EULER MACLAURIN FORMULA WITH REMAINDER FOR SIMPLE INTEGRAL POLYTOPES.

Let us begin by deriving an Euler Maclaurin formula with remainder for a simple integral orthant.

We recall the set-up of Section 5. Let \mathbf{C} be a simple integral orthant. Let u_1, \dots, u_n be the inward normals to its facets, chosen to be primitive elements of the dual lattice \mathbb{Z}^{n*} , and let $\alpha_1, \dots, \alpha_n$ be the dual basis to the u_i 's, so that

$$\mathbf{C} = v + \sum_{j=1}^n \mathbb{R}_{\geq 0} \alpha_j.$$

Let

$$\Gamma = \mathbb{Z}^{n*} / \sum \mathbb{Z} u_j$$

be the finite group associated to \mathbf{C} . By Lemma 58, $\gamma \mapsto e^{2\pi i \langle \gamma, x \rangle}$ defines a character on Γ whenever $x \in \sum \mathbb{Z} \alpha_j$, and this character is trivial if and only if $x \in \mathbb{Z}^n$. By a theorem of Frobenius, the average value of a character on a finite group is zero if the character is non-trivial and one if the character is trivial. So

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, x \rangle} = \begin{cases} 1 & \text{if } x \in \mathbb{Z}^n \\ 0 & \text{if } x \notin \mathbb{Z}^n \end{cases}$$

for all $x \in \sum \mathbb{Z}\alpha_j$. Then, for a compactly supported function $f(x)$ on \mathbb{R}^n ,

$$\begin{aligned} \sum'_{\mathbf{C} \cap \mathbb{Z}^n} f &= \sum'_x \left(\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma, x \rangle} \right) f(x) \\ (73) \qquad &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum'_x e^{2\pi i \langle \gamma, x \rangle} f(x) \end{aligned}$$

where we sum over all

$$(74) \qquad x = v + m_1 \alpha_1 + \dots + m_n \alpha_n,$$

where the m_i 's are non-negative integers.

The cone \mathbf{C} is the image of the standard orthant \mathbf{O} under the affine map

$$(75) \qquad (t_1, \dots, t_n) \mapsto x = v + t_1 \alpha_1 + \dots + t_n \alpha_n.$$

This map sends the lattice \mathbb{Z}^n onto the lattice $\sum \mathbb{Z}\alpha_j$. The inverse transformation is given by

$$(76) \qquad t_i = \langle u_i, x - v \rangle.$$

Let us concentrate on one element $\gamma \in \Gamma$. Because $v \in \mathbb{Z}^n$, from (74) we get

$$e^{2\pi i \langle \gamma, x \rangle} = \prod_{j=1}^n \lambda_j^{m_j} \quad \text{where} \quad \lambda_j = e^{2\pi i \langle \gamma, \alpha_j \rangle},$$

so that the inner sum in (73) becomes

$$(77) \qquad \sum'_{m_1 \geq 0} \lambda_1^{m_1} \dots \sum'_{m_n \geq 0} \lambda_n^{m_n} g(m_1, \dots, m_n),$$

where

$$g(t_1, \dots, t_n) = f(v + t_1 \alpha_1 + \dots + t_n \alpha_n).$$

Recall that we had the twisted remainder formula

$$\sum'_{m \geq 0} \lambda^m g(m) = \mathbf{M}^{k, \lambda} \left(\frac{\partial}{\partial h} \right) \int_{-h}^{\infty} g(t) dt \Big|_{h=0} + (-1)^{k-1} \int_0^{\infty} Q_{k, \lambda}(t) g^{(k)}(t) dt$$

for all compactly supported functions $g(x)$ of type C^k , where $k \geq 1$, where λ is a root of unity, and where $M^{k, \lambda}$ is a polynomial of degree $\leq k$. (See (28) and Remark 33.)

Iterating this formula, the sum in (77) can be written as

$$\begin{aligned} &\mathbf{M}^{k, \lambda_1} \left(\frac{\partial}{\partial h_1} \right) \int_{-h_1}^{\infty} \dots \mathbf{M}^{k, \lambda_n} \left(\frac{\partial}{\partial h_n} \right) \int_{-h_n}^{\infty} g(t_1, \dots, t_n) dt_1 \dots dt_n + R_k^{st}(\lambda_1, \dots, \lambda_n; g) \\ (78) \qquad &= \prod_{i=1}^n \mathbf{M}^{k, \lambda_i} \left(\frac{\partial}{\partial h_i} \right) \int_{\mathbf{O}(h)} g(t_1, \dots, t_n) dt_1 \dots dt_n + R_k^{st}(\lambda_1, \dots, \lambda_n; g) \end{aligned}$$

with

$$\mathbf{O}(h) = \{(t_1, \dots, t_n) \mid t_i \geq -h_i \text{ for all } i\}$$

and where the remainder is given by

$$R_k^{st}(\lambda_1, \dots, \lambda_n; g) = \sum_{I \subsetneq \{1, \dots, n\}} (-1)^{(k-1)(n-|I|)} \prod_{i \in I} \mathbf{M}^{k, \lambda_i} \left(\frac{\partial}{\partial h_i} \right) \int_{\mathbf{O}(h)} \prod_{i \notin I} Q_{k, \lambda_i}(t_i) \prod_{i \notin I} \frac{\partial^k}{\partial t_i^k} g(t_1, \dots, t_n) dt_1 \dots dt_n \Big|_{h=0}.$$

Using (46) to express this as a sum of integrals over the (non-shifted) orthant, we get

$$(79) \quad R_k^{st}(\lambda_1, \dots, \lambda_n; g) = \sum_{I \subsetneq \{1, \dots, n\}} (-1)^{(k-1)(n-|I|)} \int_{\mathbf{O}} \prod_{i \in I} \mathbf{M}^{k, \lambda_i} \left(-\frac{\partial}{\partial t_i} \right) \prod_{i \notin I} Q_{k, \lambda_i}(t_i) \prod_{i \notin I} \frac{\partial^k}{\partial t_i^k} g(t_1, \dots, t_n) dt_1 \cdots dt_n.$$

We now perform in (78) and (79) the change of variable given by the transformation (76). The integrals over $\mathbf{O}(h)$ and \mathbf{O} get replaced by integrals over $\mathbf{C}(h)$ and \mathbf{C} times the Jacobian factor $|\Gamma|$, the function $g(t)$ gets replaced by the function $f(x)$, and the partial derivative $\frac{\partial}{\partial t_i}$ gets replaced by the directional derivative D_{α_i} . Summing the expressions (78) over $\gamma \in \Gamma$ and dividing by $|\Gamma|$, we get, by (73),

$$(80) \quad \sum_{\mathbf{C} \cap \mathbb{Z}^n} 'f = \sum_{\gamma \in \Gamma} \prod_{i=1}^n \mathbf{M}^{k, \lambda_{\gamma, i}} \left(\frac{\partial}{\partial h_i} \right) \int_{\mathbf{C}(h)} f(x) dx \Big|_{h=0} + R_k^{\mathbf{C}}(f)$$

where

$$\lambda_{\gamma, j} := e^{2\pi i \langle \gamma, \alpha_j \rangle}$$

and where

$$(81) \quad R_k^{\mathbf{C}}(f) := \sum_{\gamma \in \Gamma} \sum_{I \subsetneq \{1, \dots, n\}} (-1)^{(k-1)(n-|I|)} \int_{\mathbf{C}} \prod_{i \in I} \mathbf{M}^{k, \lambda_{\gamma, i}} (-D_{\alpha_i}) \prod_{i \notin I} Q_{k, \lambda_{\gamma, i}}(\langle u_i, x - v \rangle) \prod_{i \notin I} (D_{\alpha_i})^k f(x) dx.$$

Let Δ be a simple polytope, given by (51). Choose a polarizing vector for Δ and let \mathbf{C}_v^\sharp denote the polarized tangent cones. The inward normals to the facets of \mathbf{C}_v^\sharp are given by (52), the dual basis to these vectors is $\alpha_{i,v}^\sharp$, $i \in I_v$, and the roots of unity that appear in the Euler Maclaurin formula for \mathbf{C}_v^\sharp are then

$$(82) \quad \lambda_{\gamma, i, v}^\sharp = e^{2\pi i \langle \gamma, \alpha_{i,v}^\sharp \rangle} = \begin{cases} \lambda_{\gamma, i, v} & \text{if } \alpha_{i,v}^\sharp = \alpha_{i,v} \\ \lambda_{\gamma, i, v}^{-1} & \text{if } \alpha_{i,v}^\sharp = -\alpha_{i,v}. \end{cases}$$

Also recall that the polar decomposition of $\Delta(h)$ involves the dilated orthants $C_v^\sharp(h_{i_1, v}^\sharp, \dots, h_{i_n, v}^\sharp)$ where $h_{i,v}^\sharp$ are as in (53).

Let $k \geq 1$ be an integer. For any compactly supported function f on \mathbb{R}^n of type C^{nk} , the polar decomposition of $\Delta(h)$ and the formula (80) give

$$(83) \quad \sum_{\Delta \cap \mathbb{Z}^n} 'f = \sum_v (-1)^{\#v} \sum_{\mathbf{C}_v^\sharp \cap \mathbb{Z}^n} 'f \\ = \sum_v (-1)^{\#v} \sum_{\gamma \in \Gamma_v} \prod_{i \in I_v = \{i_1, \dots, i_n\}} \mathbf{M}^{k, \lambda_{\gamma, i, v}^\sharp} \left(\frac{\partial}{\partial h_{i,v}^\sharp} \right) \int_{\mathbf{C}_v^\sharp(h_{i_1, v}^\sharp, \dots, h_{i_n, v}^\sharp)} f(x) dx \Big|_{h=0} + R_k^\Delta(f),$$

where the remainder is given by

$$(84) \quad R_k^\Delta(f) := \sum_v (-1)^{\#v} R_k^{\mathbf{C}_v^\sharp}(f).$$

Note that either $h_{i,v}^\# = h_i$ and $\lambda_{\gamma,i,v}^\# = \lambda_{\gamma,i,v}$, or $h_{i,v}^\# = -h_i$ and $\lambda_{\gamma,i,v}^\# = \lambda_{\gamma,i,v}^{-1}$. By the symmetry property (30), this gives

$$\mathbf{M}^{k,\lambda_{\gamma,i,v}^\#} \left(\frac{\partial}{\partial h_{i,v}^\#} \right) = \mathbf{M}^{k,\lambda_{\gamma,i,v}} \left(\frac{\partial}{\partial h_i} \right).$$

For $j \notin I_v$, because $\lambda_{\gamma,j,v} = 1$ (see (70)), we have $\mathbf{M}^{k,\lambda_{\gamma,j,v}} \left(\frac{\partial}{\partial h_j} \right) = 1 + \text{powers of } \frac{\partial}{\partial h_j}$. Also, the cone $C_v^\#(h_{i_1,v}^\#, \dots, h_{i_n,v}^\#)$ is independent of h_j for $j \notin I_v$. Therefore, (83) is further equal to

$$(85) \quad \sum_v (-1)^{\#v} \sum_{\gamma \in \Gamma_v} \prod_{j=1}^d \mathbf{M}^{k,\lambda_{\gamma,j,v}} \left(\frac{\partial}{\partial h_j} \right) \int_{C_v^\#(h_{i_1,v}^\#, \dots, h_{i_n,v}^\#)} f(x) dx \Big|_{h=0} + R_k^\Delta(f)$$

where in the v 'th summand $\{i_1, \dots, i_n\} = I_v$.

Define

$$(86) \quad \mathbf{M}_{\gamma,F}^k = \prod_{j=1}^d \mathbf{M}^{k,\lambda_{\gamma,j,F}} \left(\frac{\partial}{\partial h_j} \right) \quad \text{for } \gamma \in \Gamma_F.$$

Then we have, by (69),

$$(87) \quad \mathbf{M}_{\gamma,v}^k = \mathbf{M}_{\gamma,v}^k \quad \text{whenever } \gamma \in \Gamma_F \text{ and } v \in F,$$

where we identify $\gamma \in \Gamma_F$ with its image under the inclusion map $\Gamma_F \hookrightarrow \Gamma_v$.

Then (85) is equal to

$$(88) \quad \begin{aligned} & \sum_v (-1)^{\#v} \sum_{\gamma \in \Gamma_v} \mathbf{M}_{\gamma,v}^k \int_{C_v^\#(h_{i_1,v}^\#, \dots, h_{i_n,v}^\#)} f(x) dx \Big|_{h=0} + R_k^\Delta(f) \\ &= \sum_F \sum_{\gamma \in \Gamma_F^b} \mathbf{M}_{\gamma,F}^k \sum_{v \in F} (-1)^{\#v} \int_{C_v^\#(h_{i_1,v}^\#, \dots, h_{i_n,v}^\#)} f(x) dx \Big|_{h=0} + R_k^\Delta(f), \end{aligned}$$

by (65) and (87). In the interior summation we may now add similar summands that correspond to $v \notin F$. These summands make a zero contribution to (88) for the following reason. If $v \notin F$ then there exists $i \in I_F \setminus I_v$. Because $i \notin I_v = \{i_1, \dots, i_n\}$, the cone $C_v^\#(h_{i_1,v}^\#, \dots, h_{i_n,v}^\#)$ is independent of h_i . So it is enough to show that $\mathbf{M}_{\gamma,F}^k$ is a multiple of $\frac{\partial}{\partial h_i}$. But because $\gamma \in \Gamma_F^b$ and $i \in I_F$, we have $\lambda_{\gamma,i,F} \neq 1$. (See Claim 71.) By Remark 33, this implies that $\mathbf{M}^{k,\lambda_{\gamma,i,F}} \left(\frac{\partial}{\partial h_i} \right)$, which is one of the factors in $\mathbf{M}_{\gamma,F}^k$, is a multiple of $\frac{\partial}{\partial h_i}$. Hence, (88) is equal to

$$(89) \quad \begin{aligned} & \sum_F \sum_{\gamma \in \Gamma_F^b} \mathbf{M}_{\gamma,F}^k \sum_{\text{all } v} (-1)^{\#v} \int_{C_v^\#(h_{i_1,v}^\#, \dots, h_{i_n,v}^\#)} f(x) dx \Big|_{h=0} + R_k^\Delta(f) \\ &= \sum_F \sum_{\gamma \in \Gamma_F^b} \mathbf{M}_{\gamma,F}^k \int_{\Delta(h)} f(x) dx \Big|_{h=0} + R_k^\Delta(f). \end{aligned}$$

We have therefore proved our main result:

Theorem 2. *Let Δ be a simple integral polytope in \mathbb{R}^n and let $f \in C_c^{nk}(\mathbb{R}^n)$ be a compactly supported function on \mathbb{R}^n for $k \geq 1$. Choose a polarizing vector for Δ . Then*

$$\sum_{\Delta \cap \mathbb{Z}^n} 'f = \sum_F \sum_{\gamma \in \Gamma_F^b} \mathbf{M}_{\gamma, F}^k \int_{\Delta(h)} f(x) dx \Big|_{h=0} + R_k^\Delta(f)$$

where $\mathbf{M}_{\gamma, F}^k$ are differential operators defined in (86) and where the remainder $R_k^\Delta(f)$ is given by equation (84). Moreover, the differential operators $\mathbf{M}_{\gamma, F}^k$ are of order $\leq k$ in each of the variables h_1, \dots, h_d . Also, the remainder is a sum of integrals over orthants of bounded periodic functions times various partial derivatives of f of order no less than k and no more than kn . Finally, this remainder is independent of the choice of polarization and is a distribution supported on the polytope Δ .

7. ESTIMATES ON THE REMAINDER AND EULER MACLAURIN FORMULAS FOR SYMBOLS AND FOR POLYNOMIALS

We first recall a definition from the theory of partial differential equations. A smooth function $f \in C^\infty(\mathbb{R}^n)$ is called a **symbol of order N** if for every n -tuple of non-negative integers $a := (a_1, \dots, a_n)$, there exists a constant C_a such that

$$|\partial_1^{a_1} \dots \partial_n^{a_n} f(x)| \leq C_a (1 + |x|)^{N - |a|}$$

where $|a| = \sum_i a_i$. In particular, a polynomial of degree N is a symbol of order N . Note that if f is a symbol of order N on \mathbb{R}^n then its derivatives of order a are in L^1 if $N < |a| - n$. In this section we will show that the Euler Maclaurin formula of Theorem 2 can be extended to symbols, and gives rise in this way to an *exact* Euler Maclaurin formula for polynomials.

To make further progress, we first require an estimate on the remainder term $R_k^\Delta(f)$. We recall (see Theorem 2) that this remainder can be expressed as a sum of integrals over orthants of bounded periodic functions times various partial derivatives of f of order no less than k and no more than kn . Explicitly, from (84) and (81) we get

$$(90) \quad R_k^\Delta(f) = \sum_v (-1)^{\#v} \sum_{\gamma \in \Gamma_v} \sum_{I \subsetneq I_v} (-1)^{(k-1)(n-|I|)} \int_{\mathbf{C}_v^\#} \prod_{j \in I} \mathbf{M}^{k, \lambda_{\gamma, j, v}^\#} (-D_{\alpha_{j, v}^\#}) \prod_{j \notin I} Q_{k, \lambda_{\gamma, j, v}^\#} (\langle u_{j, v}^\#, x - v \rangle) \prod_{j \notin I} (D_{\alpha_{j, v}^\#})^k f(x) dx.$$

The functions $Q_{k, \lambda}$ are bounded and periodic, and $M^{k, \lambda}(-D_{\alpha_j})$ are differential operators of order k . In particular, each integrand in the formula for $R_k^\Delta(f)$ is dominated by a constant times

$$\sup_{\{j_1, \dots, j_n\}} |\partial_1^{j_1} \dots \partial_n^{j_n} f|$$

where the supremum is taken over all n -tuples $\{j_1, \dots, j_n\}$ with $k \leq j_1 + \dots + j_n \leq nk$. Consequently, $R_k^\Delta(f)$ is well defined (by the same formula (90)) for any smooth function f whose derivatives of order between k and nk are integrable on \mathbb{R}^n . In particular, it is defined when f is a symbol of order less than $k - n$. Moreover, we get an estimate for the remainder:

Proposition 91. *The distribution $R_k^\Delta(\cdot)$ extends to symbols of order less than $k - n$, by the explicit expression (90) above, and satisfies an estimate of the form*

$$|R_k^\Delta(f)| \leq K(k, \Delta) \cdot \sup_{\{j_1, \dots, j_n\}} |\partial_1^{j_1} \dots \partial_n^{j_n} f|_{L^1(\mathbb{R}^n)},$$

where the supremum is taken over all n -tuples $\{j_1, \dots, j_n\}$ with $k \leq j_1 + \dots + j_n \leq nk$.

By applying this estimate we will obtain the following Euler Maclaurin formula for symbols.

Theorem 3. *Let Δ be a simple integral polytope in \mathbb{R}^n , let f be a symbol of order N on \mathbb{R}^n , and choose $k \geq N + n + 1$. Then*

$$(92) \quad \sum_{\Delta \cap \mathbb{Z}^n} 'f = \sum_F \sum_{\gamma \in \Gamma_F^b} \mathbf{M}_{\gamma, F}^k \int_{\Delta(h)} f(x) dx \Big|_{h=0} + R_k^\Delta(f)$$

where $\mathbf{M}_{\gamma, F}^k$ are differential operators defined in (86) and where the remainder term $R_k^\Delta(f)$ is defined in (90). Moreover, the differential operators $\mathbf{M}_{\gamma, F}^k$ are of order $\leq k$ in each of the variables h_1, \dots, h_d . Also, the remainder $R_k^\Delta(f)$ satisfies the estimate of Proposition 91.

In the case where f is a polynomial, this formula gives rise to an *exact* Euler Maclaurin formula.

Corollary 93. Let p be a polynomial on \mathbb{R}^n , and choose $k \geq \deg p + n + 1$. Then

$$\sum_{\Delta \cap \mathbb{Z}^n} 'p = \sum_F \sum_{\gamma \in \Gamma_F^b} \mathbf{M}_{\gamma, F}^k \int_{\Delta(h)} p(x) dx \Big|_{h=0}.$$

Proof of Theorem 3: Let χ be a smooth function on \mathbb{R}^n which is equal to one on some open ball about the origin that contains Δ and which is supported in some larger ball, say, of radius R . Define $\chi_\lambda(x) := \chi(x/\lambda)$ for all $\lambda \geq 1$. Then the function

$$f_\lambda := f \chi_\lambda$$

is a smooth compactly supported function on \mathbb{R}^n . We apply Theorem 2 to obtain

$$(94) \quad \sum_{\Delta \cap \mathbb{Z}^n} 'f_\lambda = \sum_F \sum_{\gamma \in \Gamma_F^b} \mathbf{M}_{\gamma, F}^k \int_{\Delta(h)} f_\lambda(x) dx \Big|_{h=0} + R_k^\Delta(f_\lambda),$$

which is valid for any $k \geq 1$.

Since f_λ equals f on a neighborhood of Δ if $\lambda \geq 1$, the left hand side and the first of the two summands on the right hand side in (94) are equal to the corresponding terms in (92). Thus, to deduce (92) from (94), it suffices to prove the following claim:

$$(95) \quad \lim_{\lambda \rightarrow \infty} R_k^\Delta(f_\lambda) = R_k^\Delta(f) \quad \text{if} \quad k \geq N + n + 1.$$

To prove this claim, we apply the estimate in Proposition 91 to the difference $R_k^\Delta(f) - R_k^\Delta(f_\lambda) = R_k^\Delta(f(1 - \chi_\lambda))$. We expand each of the derivatives appearing in the estimate into a finite sum of products of derivatives of f and derivatives of $1 - \chi_\lambda$. The leading term has the form $g(1 - \chi_\lambda)$ where g is a derivative of f of order q with $q \geq k$. Because f is a symbol of order N , the function g is dominated by a constant times the function $x \mapsto (1 + |x|)^{N-q}$. Because $q \geq k \geq N + n + 1$, the function g is in L_1 . It follows that the L_1 norm of $g(1 - \chi_\lambda)$ converges to zero as $\lambda \rightarrow \infty$. Each of the remaining terms in the expansion has the form

$$(96) \quad \lambda^{-s} g(\cdot) \tilde{\chi}(\cdot/\lambda)$$

where g is a derivative of f of order q and $\tilde{\chi}$ is a derivative of $1 - \chi$ of order s , and where $s \geq 1$ and $s + q \geq k$. The function g is dominated by a constant times the function $x \mapsto (1 + |x|)^{N-q}$; the function $\tilde{\chi}$ is bounded; the function $\tilde{\chi}(\cdot/\lambda)$ is supported on the ball of radius λR about the origin. Hence, the L_1 norm of the product $g(\cdot) \tilde{\chi}(\cdot/\lambda)$ is bounded by a constant times λ^{N-q+n} if $N - q + n$ is non-negative and by a constant otherwise. Since $s \geq 1$ and $s + q \geq k \geq N + n + 1$, the L_1 norm of the term (96) is bounded by some constant multiple of λ^{-1} . Letting $\lambda \rightarrow \infty$, we see that each of the terms in the estimate approaches zero as $\lambda \rightarrow \infty$. This implies (95). Theorem 3 follows. \square

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INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL, AND: DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 3G3, CANADA

E-mail address: karshon@math.toronto.edu

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA

E-mail address: shlomo@math.harvard.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CA 95064, USA

E-mail address: weitsman@math.ucsc.edu